

MAS 117

Introduction to Algebra

Assignment 1

For handing in on 17 January 2007

*Write your name and student number at the top of your assignment before handing it in. Staple all the pages together. Post the assignment in the **RED** post-box on the **BASEMENT** of the Maths building before 16:00 on Wednesday.*

- 1** (20 marks) When the mathematician Gauss was in primary school, his teacher asked the class to add up all the numbers from 1 to 100. Gauss saw that, if he took the sum

$$S = 1 + 2 + 3 + \cdots + 99 + 100,$$

and wrote it down reversed,

$$S = 100 + 99 + 98 + \cdots + 2 + 1,$$

then each pair of numbers in the two sums adds up to 101. So

$$2S = 100 \times 101 = 10100,$$

and $S = 5050$. Your job is to turn this into a general proof that the sum of the natural numbers from 1 to n is $n(n+1)/2$.

- 2** (20 marks) Find a counterexample to the statement:

If p is prime, then $2^p - 1$ is prime.

You should explain what a counterexample to this statement is.

- 3** (20 marks) Is the following argument valid? If not, why not?

We are going to prove that a triangle whose sides have lengths 3, 4 and 5 is right-angled.

By Pythagoras' Theorem, if a triangle with sides a, b, c is right-angled, with hypotenuse c , then $a^2 + b^2 = c^2$.

Now $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

So the triangle is right-angled.

4 (15 marks) Let $a_1, a_2, a_3, \dots$ be numbers satisfying the rules that $a_1 = 1$ and $a_n = 2a_{n-1}$ for all $n > 1$.

- (a) Write down the first few numbers a_n .
- (b) Guess a formula for a_n .
- (c) Prove your guess by induction.

5 (25 marks) Is the following argument valid? If not, why not?

We are going to prove that all horses have the same colour. To be more precise, let $H_1, H_2, \dots, H_n$ be horses; we will prove, by induction on n , that $H_1, H_2, \dots, H_n$ all have the same colour.

We begin the induction with the case $n = 1$. There is only one horse, so clearly the statement is true.

Now for the induction step. Suppose that, given any $n - 1$ horses, they all have the same colour. Suppose we are given n horses $H_1, H_2, \dots, H_n$. Then by the induction hypothesis,

- $H_1, \dots, H_{n-1}$ all have the same colour;
- $H_2, \dots, H_n$ all have the same colour.

So all of $H_1, \dots, H_n$ have the same colour, and the induction step is proved.

So the assertion is proved by induction.

6 Prize question A cash prize is offered for the best solution received to this question. Do not hand this in with the rest of your coursework; give your solutions to me.

Euclid's proof that there are infinitely many primes gives us a rule for finding a new prime, if we already have a finite number:

Suppose that we have found n primes already, say $p_1, p_2, \dots, p_n$.

Multiply them together and add one: let N be this number, so that $N = p_1 p_2 \cdots p_n + 1$.

If N is prime, take it to be the next prime p_{n+1} . Otherwise, take p_{n+1} to be the smallest prime which divides N .

Euclid gives us a guarantee that p_{n+1} is different from all the primes $p_1, \dots, p_n$.

Take $p_1 = 2$. Use MAPLE or a calculator to find $p_2, p_3, \dots, p_8$.

Experiment with taking different primes for p_1 . Does the prime 2 always turn up in the list sooner or later? Does the prime 3 always turn up? What is the main difficulty in the calculation?